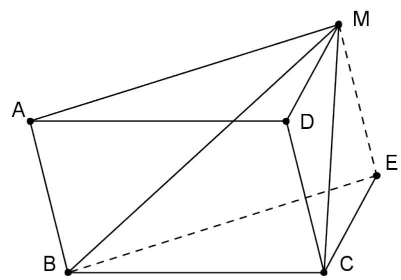
Călăraşi, 2011

1. Let $ABCM$ be a quadrilateral and D be an interior point such that $ABCD$ is a parallelogram. It is known that $\angle AMB \equiv \angle CMD$. Prove that $\angle MAD \equiv \angle MCD$.



Solution. Construct parallelogram $ABEM$. Then $\angle AMB \equiv \angle MBE$ and, since $CDME$ is a parallelogram, $\angle DMC \equiv \angle MCE$. This leads to $\angle MBE \equiv \angle MCE$, so the quadrilateral $MBCE$ is cyclic. This yields $\angle BCM \equiv \angle BEM \equiv \angle BAM$, whence the conclusion.

2. Let S be a set of positive integer numbers such that

$$\min\{\operatorname{lcm}(x, y) : x, y \in S, x \neq y\} \geq 2 + \max S.$$

Show that

$$\sum_{x \in S} 1/x < 3/2.$$

Solution. The condition in the statement implies that there exists a positive integer n which is a strict upper bound for S and a strict lower bound for the set of the least common multiples of distinct numbers in S. If x is a member of S, let M_x denote the set of positive multiples of x that do not exceed n. Clearly, $|M_x| = \lfloor n/x \rfloor$. If x and y are distinct members of S, then M_x and M_y are disjoint, for the least common multiple of x and y is greater than n. Consequently,

$$\sum_{x \in S} \lfloor n/x \rfloor = \sum_{x \in S} |M_x| \leq n$$

and $|S| \leq \lfloor n/2 \rfloor$ (otherwise, some number in S would divide another, by a well-known result of Erdős). Finally,

$$n \sum_{x \in S} 1/x - n/2 \leq n \sum_{x \in S} 1/x - |S| = \sum_{x \in S} (n/x - 1) < \sum_{x \in S} \lfloor n/x \rfloor \leq n,$$

whence the conclusion.

3. Determine all positive integer numbers n satisfying the following condition: the sum of the squares of any n prime numbers greater than 3 is divisible by n.

Solution. We begin by showing that if a positive integer k is relatively prime to n, then $k^2 \equiv 1 \pmod{n}$. To this end, invoke the Dirichlet theorem on arithmetic sequences to choose $n-1$ primes congruent to 1 modulo n and a prime congruent to k modulo n. The sum of the squares of these n primes is congruent to $k^2 - 1$ modulo n, so $k^2 \equiv 1 \pmod{n}$.

Next, we prove that n has no prime divisors greater than 3. Let p be an odd divisor of n and write $n = p^\alpha m$, where α and m are positive integers and p does not divide m. By the Chinese Remainder Theorem, there exists a positive integer k such that $k \equiv 1 \pmod{m}$ and $k \equiv 2 \pmod{p}$. It is easily seen that k and n are coprime, so $k^2 \equiv 1 \pmod{n}$ by the preceding. Hence $k^2 \equiv 1 \pmod{p}$, and the condition $k \equiv 2 \pmod{p}$ forces $p = 3$.

Consequently, $n = 2^\alpha 3^\beta$, where α and β are non-negative integers. Since 5 and n are coprime, the latter must be a divisor of $5^2 - 1 = 24$. It is readily checked that all divisors of 24 work.

4. Given a positive integer number n, determine the maximum number of edges a triangle-free Hamiltonian simple graph on n vertices may have.

Solution. The required maximum is $\lfloor n/2 \rfloor^2$ if n is even and $\lfloor n/2 \rfloor^2 + 1$ if n is odd.

By Turán's theorem, the maximum number of edges a triangle-free simple graph on n vertices may have is $\lfloor n/2 \rfloor \lfloor (n+1)/2 \rfloor$ and is achieved only by the complete graph $K_{\lfloor n/2 \rfloor, \lfloor (n+1)/2 \rfloor}$. If n is even, the latter is also Hamiltonian and we are done.

Consider the case $n = 2m + 1$, where m is an integer number greater than 1. Let G be a triangle-free Hamiltonian simple graph on n vertices with a maximum number of edges. Since G is Hamiltonian and has an odd number of vertices, it has an odd cycle, so it must have a shortest odd cycle, say C, of length $2k + 1$, where k is an integer number greater than 1. No additional edges forming diagonals in C may exist without creating a shorter odd cycle. Each of the $2m - 2k$ vertices outside C may be joined to at most two vertices of C, for any choice of more vertices of C would yield a shorter odd cycle. Finally, with reference again to Turán's theorem, the $2m - 2k$ vertices outside C may induce at most $(m - k)^2$ edges without forming any triangles. Consequently, G has at most

$$(2k + 1) + 2(2m - 2k) + (m - k)^2$$

edges. The largest possible value occurs when $k = 2$ and is $m^2 + 1$. Many Hamiltonian graphs achieve this bound. One of them, H_n, is constructed from $K_{m,m}$ by inserting a vertex of degree 2 on any one edge.